Mathematics
Higher Level
Paper 2

Name

Date: ____________________

2 hours

SOLUTION KEY

Instructions to candidates

- Write your name in the box above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Section A: answer all of Section A in the spaces provided.
- Section B: answer all of Section B on the answer sheets provided. Write your name on each answer sheet and attach them to this examination paper.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

worked solutions: 19 pages

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, for example if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A (57 marks)

Answer **all** questions in the boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 6]

A study is conducted to compare the monthly e-commerce sales of nine separate online stores to their monthly online advertising costs. The table below shows, in units of $1000, the monthly e-commerce sales (y) of each online store and their monthly online advertising costs (x).

The relationship between the monthly e-commerce sales and the monthly online advertising costs can be modelled by the regression line with equation $y = ax + b$.

Online Advertising Costs (x)	1.4	1.7	2.3	1.1	4.7	2.2	2.9	3.8	1.9
E-Commerce Sales (y)	343	371	587	320	921	492	646	835	413

(a) (i) Find Pearson's product moment correlation coefficient, r.

(ii) Write down the value of a and the value of b. [3]

One of these nine online stores decides to increase their budget for monthly online advertising costs by $500.

(b) Based on the given data, determine how the store's monthly e-commerce sales could be expected to alter. [2]

An online store separate from the study has monthly online advertising costs of $7000.

(c) Comment on the appropriateness of using your regression line to predict the monthly e-commerce sales of this separate online store. [1]

(This question continues on the following page)

(Question 1 continued)

(a) (i) Using GDC, $r \approx 0.985$

(ii) Using GDC, $a \approx 183,\ b \approx 99.6$

(b) finding the difference in y for x and $x+0.5$:

$$\Delta y = (183.26\ldots)(x+0.5)+99.577\ldots-\left[(183.26\ldots)x+99.577\ldots\right]$$

$$\Delta y = 0.5\cdot(183.26\ldots)$$

$$\Delta y = 91.631\ldots$$

Thus, the online store can expect an increase in monthly e-commerce sales by $91,600

(c) This is extrapolation which is not appropriate.

2. [Maximum mark: 6]

Triangle FGH has $\mathrm{FG} = 8 \text{ cm}$, $\mathrm{GH} = 9 \text{ cm}$ and area 24 cm^2.

(a) Find $\sin\hat{G}$. [2]

(b) Hence, find the two possible values of FH, giving your answers correct to two decimal places. [4]

(a) $\text{area} = \frac{1}{2} \cdot FG \cdot GH \cdot \sin G$

$24 = \frac{1}{2} \cdot 8 \cdot 9 \sin G$

$\sin G = \frac{24}{36} = \frac{2}{3}$

(b) $\sin^2 G + \cos^2 G = 1$

$\cos^2 G = 1 - \frac{4}{9} = \frac{5}{9}$

$\cos G = \frac{\sqrt{5}}{3}$ or $\cos G = -\frac{\sqrt{5}}{3}$

$G = \cos^{-1}\left(\frac{\sqrt{5}}{3}\right) \approx 41.803°$ or $G = \cos^{-1}\left(-\frac{\sqrt{5}}{3}\right) \approx 138.19°$

$FH^2 = FG^2 + GH^2 - 2(FG)(GH)\cos G$

$FH = \sqrt{8^2 + 9^2 - 2(8)(9)\cos(41.803...°)}$

$FH \approx 6.14 \text{ cm}$

OR $FH = \sqrt{8^2 + 9^2 - 2(8)(9)\cos(138.19...°)}$

$FH \approx 15.9 \text{ cm}$

3. [Maximum mark: 6]

The sum of the first n terms of a series is given by

$$S_n = 3n^2 + n,\ n \in \mathbb{Z}^+$$

(a) Find the first three terms of the series. [3]

(b) Find an expression for the n^{th} term of the series, giving your answer in terms of n. [3]

(a) $S_n = 3n^2 + n$

$n=1$: $S_1 = u_1 = 3 + 1 = 4$ $\quad u_1 = 4$

$n=2$: $S_2 = u_1 + u_2 = 12 + 2 = 14$

$u_2 = 14 - u_1 = 14 - 4 = 10$ $\quad u_2 = 10$

$n=3$: $S_3 = u_1 + u_2 + u_3 = 27 + 3 = 30$

$u_3 = 30 - u_1 - u_2 = 30 - 4 - 10 = 16$ $\quad u_3 = 16$

(b) $u_n = S_n - S_{n-1}$

$u_n = 3n^2 + n - \left[3(n-1)^2 + n - 1\right]$

$= 3n^2 + n - 3(n^2 - 2n + 1) - n + 1$

$u_n = 6n - 2$

4. [Maximum mark: 6]

Using the substitution $2x = \sin\theta$, or otherwise, find $\int\left(\sqrt{1-4x^2}\right)dx$.

$$2x = \sin\theta \Rightarrow 2\frac{dx}{d\theta} = \cos\theta$$

$$dx = \frac{1}{2}\cos\theta \, d\theta$$

$$\int\left(\sqrt{1-4x^2}\right)dx = \int\left(\sqrt{1-\sin^2\theta}\right)\left(\frac{1}{2}\cos\theta\right)d\theta$$

$$= \frac{1}{2}\int \cos^2\theta \, d\theta$$

$$2\cos^2\theta - 1 = \cos 2\theta \Rightarrow \cos^2\theta = \frac{1}{2}\left(\cos 2\theta + 1\right)$$

$$\int\left(\sqrt{1-4x^2}\right)dx = \frac{1}{2}\int \frac{1}{2}\left(\cos 2\theta + 1\right)d\theta$$

$$= \frac{1}{4}\int\left(\cos 2\theta + 1\right)d\theta$$

$$= \frac{1}{4}\left(\frac{1}{2}\sin 2\theta + \theta\right) + C$$

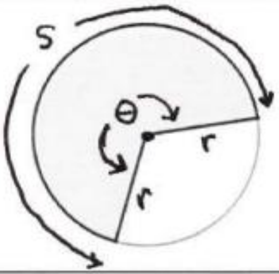
$$2x = \sin\theta \rightarrow \theta = \arcsin(2x)$$

$$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-(2x)^2} = \sqrt{1-4x^2}$$

substituting gives $\int\left(\sqrt{1-4x^2}\right)dx = \frac{1}{4}\left[2x\sqrt{1-4x^2} + \arcsin(2x)\right] + C$

5. [Maximum mark: 6]

In the figure below, the shaded sector has a perimeter that is equal to the circumference of the circle. The location of a point inside the circle is chosen at random. Find the probability that the randomly chosen point is located inside the shaded sector.



$$\text{probability point in shaded sector} = \frac{\text{area shaded sector}}{\text{area of circle}}$$

let θ = angle of shaded sector,

r = radius of circle, and s = length of arc of shaded sector

$$\text{perimeter shaded sector} = 2r + s$$
$$= 2r + \theta r$$

solve for θ: $\theta = 2\pi - 2$

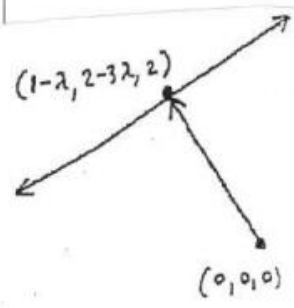
$$\frac{\text{area of shaded sector}}{\text{area of circle}} = \frac{\frac{1}{2}\theta r^2}{\pi r^2} = \frac{\theta}{2\pi}$$

$$\text{substituting} \quad = \frac{2\pi - 2}{2\pi}$$

$$\text{probability} = \frac{\pi - 1}{\pi}$$

6. [Maximum mark: 7]

The line L is given by the parametric equations $x = 1 - \lambda,\ y = 2 - 3\lambda,\ z = 2$. Find the coordinates of the point on L which is nearest to the origin.



the position vector for the point nearest to the origin is perpendicular to the direction of the line

for line L : $\left.\begin{array}{l} x = 1-\lambda \\ y = 2-3\lambda \\ z = 2 \end{array}\right\} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -3 \\ 0 \end{pmatrix}$

vector eq. for line L

hence, a direction vector for line L is $\begin{pmatrix} -1 \\ -3 \\ 0 \end{pmatrix}$

and position vector of any point on line L is $\begin{pmatrix} 1-\lambda \\ 2-3\lambda \\ 2 \end{pmatrix}$

thus, $\begin{pmatrix} -1 \\ -3 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1-\lambda \\ 2-3\lambda \\ 2 \end{pmatrix} = 0 \Rightarrow -(1-\lambda) - 3(2-3\lambda) = 0$

$-1 + \lambda - 6 + 9\lambda = 0$

$10\lambda = 7 \Rightarrow \underline{\lambda = \frac{7}{10}}$

$\underline{\text{for } \lambda = \frac{7}{10}}$: $x = 1 - \frac{7}{10} = \frac{3}{10}$, $y = 2 - 3\left(\frac{7}{10}\right) = -\frac{1}{10}$, $z = 2$

$\therefore$ point of intersection is $\underline{\left(\frac{3}{10}, -\frac{1}{10}, 2\right)}$

7. [Maximum mark: 6]

Consider the function $f(x)=\ln\left(1-x^2\right)$.

(a) (i) Determine the domain of $f(x)$.

(ii) Find the first three terms in the Maclaurin series for $f(x)$. [3]

(b) Hence, show that the exact value of $\sum_{n=1}^{\infty}\frac{2^{-2n}}{n}$ is $\ln\left(\frac{4}{3}\right)$. [3]

(a) (i) $1-x^2>0 \Rightarrow x^2>1 \Rightarrow -1<x<1$

(ii) from formula booklet, the Maclaurin series for $\ln(1+x)$ is:

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots$$

Substituting to find the Maclaurin series for $\ln\left(1-x^2\right)$:

$$\ln\left(1-x^2\right)=\ln\left(1+\left(-x^2\right)\right)=\left(-x^2\right)-\frac{\left(-x^2\right)^2}{2}+\frac{\left(-x^2\right)^3}{3}-\ldots$$

$$\ln\left(1-x^2\right)=-x^2-\frac{x^4}{2}-\frac{x^6}{3}-\ldots$$

Thus, the first three terms in the Maclaurin series for $f(x)$ are $-x^2$, $-\frac{x^4}{2}$ and $-\frac{x^6}{3}$

(b) $$\sum_{n=1}^{\infty}\frac{2^{-2n}}{n}=\sum_{n=1}^{\infty}\frac{1}{n}\cdot\left(\frac{1}{2}\right)^{2n}=\frac{1}{1}\cdot\left(\frac{1}{2}\right)^2+\frac{1}{2}\cdot\left(\frac{1}{2}\right)^4+\frac{1}{3}\cdot\left(\frac{1}{2}\right)^6+\ldots$$

Substituting $x=\frac{1}{2}$ into the Maclaurin series for $\ln\left(1-x^2\right)$:

$$\ln\left(1-\left(\frac{1}{2}\right)^2\right)=-\left(\frac{1}{2}\right)^2-\frac{1}{2}\cdot\left(\frac{1}{2}\right)^4-\frac{1}{3}\cdot\left(\frac{1}{2}\right)^6-\ldots$$

So, $$\sum_{n=1}^{\infty}\frac{2^{-2n}}{n}=-\ln\left(1-\left(\frac{1}{2}\right)^2\right)=-\ln\left(1-\frac{1}{4}\right)=-\ln\left(\frac{3}{4}\right)\quad\left[=\ln\left(\frac{4}{3}\right)\right]$$ ***Q.E.D.***

8. [Maximum mark: 8]

If α and β are the roots of the equation $2x^2+6x-5=0$, find a quadratic equation with integer coefficients whose roots are:

(a) 2α, 2β [4]

(b) $\dfrac{1}{\alpha+1}, \dfrac{1}{\beta+1}$ [4]

For the equation $2x^2+6x-5=0$, $a=2$, $b=6$ and $c=-5$.

Thus, $\alpha+\beta=-\frac{b}{a}=-\frac{6}{2}=-3$ and $\alpha\beta=\frac{c}{a}=\frac{-5}{2}$.

a) Sum of the new roots $=2\alpha+2\beta=2(\alpha+\beta)=2(-3)=-6$.

Thus for the new equation, $-\frac{b}{a}=-6$.

Product of the new roots $=2\alpha\cdot 2\beta=4\alpha\beta=4\left(-\frac{5}{2}\right)=-10$.

Thus for the new equation, $\frac{c}{a}=-10$.

The new equation we are looking for can be written as $ax^2+bx+c=0$ or $x^2+\frac{b}{a}x+\frac{c}{a}=0$.

Therefore, the quadratic equation with roots $2\alpha, 2\beta$ is $x^2-(-6)x-10=0$
$\Rightarrow x^2+6x-10=0$

b) Sum of the new roots $\dfrac{1}{\alpha+1}+\dfrac{1}{\beta+1}=\dfrac{\beta+1+\alpha+1}{(\alpha+1)(\beta+1)}$
$=\dfrac{\alpha+\beta+2}{\alpha\beta+\alpha+\beta+1}=\dfrac{-3+2}{-\frac{5}{2}-3+1}=\dfrac{-1}{-\frac{9}{2}}=\dfrac{2}{9}$.

Thus for the new equation, $-\frac{b}{a}=\frac{2}{9}$.

Product of the new roots $\left(\dfrac{1}{\alpha+1}\right)\left(\dfrac{1}{\beta+1}\right)=\dfrac{1}{\alpha\beta+\alpha+\beta+1}$
$=\dfrac{1}{-\frac{5}{2}-3+1}=\dfrac{1}{-\frac{9}{2}}=-\dfrac{2}{9}$.

Thus for the new equation, $\frac{c}{a}=-\frac{2}{9}$.

The new equation we are looking for can be written as $x^2+\frac{b}{a}x+\frac{c}{a}=0$.

Therefore, the quadratic equation with roots
$\dfrac{1}{\alpha+1}, \dfrac{1}{\beta+1}$ is $x^2-\frac{2}{9}x-\frac{2}{9}=0$ or $9x^2-2x-2=0$.

9. [Maximum mark: 6]

There are 25 marbles in a bag. Some of them yellow and the rest are blue. Two marbles are simultaneously selected at random. Given that the probability of selecting two marbles of the same colour is equal to the probability of selecting two marbles of different colour, determine the number of yellow marbles in the bag.

let n = # of yellow marbles, and $25-n$ = # of blue marbles

$$P(\text{same colour}) = \frac{n(n-1)}{25\cdot 24} + \frac{(25-n)(24-n)}{25\cdot 24}$$

$$P(\text{different colour}) = \frac{n(25-n)}{25\cdot 24} + \frac{(25-n)n}{25\cdot 24}$$

setting the two probabilities equal

$$\frac{n^2-n}{600} + \frac{600-49n+n^2}{600} = \frac{25n-n^2}{600} + \frac{25n-n^2}{600}$$

$$2n^2-50n+600 = -2n^2+50n$$

$$4n^2-100n+600 = 0$$

$$n^2-25n+150 = 0$$

$$(n-10)(n-15) = 0$$

$$n = 10 \quad \text{or} \quad n = 15$$

# of yellow marbles is 10 or 15

Do **not** write solutions on this page.

Section B (53 marks)

Answer **all** the questions on the answer sheets provided. Please start each question on a new page.

10. [Maximum mark: 21]

Consider the quadratic function defined by $f(x) = x^2 - 7x + 13$ with domain $x \geq \frac{7}{2}$.

(a) Sketch the graph of f. [2]

(b) State the range of f. [2]

(c) Find the inverse of function of f, that is find $f^{-1}(x)$. [5]

(d) Write down the domain and range of $f^{-1}(x)$. [2]

(e) Find all values of x such that $f(x) = f^{-1}(x)$. [3]

Consider the function $g(x) = \dfrac{1}{x-1}$.

(f) Sketch the graph of $g(f(x))$, clearly indicating any asymptotes. [3]

(g) Determine the domain of $g(f(x))$. [2]

(h) Determine the range of $g(f(x))$. [2]

Do **not** write solutions on this page.

11. [Maximum mark: 16]

It has been determined that the volume of fluid in a bottle of olive oil filled by a robotic dispenser in a factory is normally distributed with a mean of 748 ml and a standard deviation of 2.4 ml.

(a) Show that the probability that a randomly selected bottle of olive oil from the factory contains more than 750 ml is approximately 0.202 [1]

(b) The amount of olive oil is measured for each bottle in a random sample of 12 bottles. Find the probability that exactly 4 of them contain more than 750 ml. [3]

(c) Find the minimum number of bottles that would need to be sampled so that the probability of getting at least one bottle containing more than 750 ml of olive oil is greater than 0.98. [3]

The same factory produces bags of flour, such that the weight, A grams, of flour in a bag is normally distributed with mean μ grams and standard deviation σ grams.

(d) Given that $\mathrm{P}(A<850)=0.09$ and $\mathrm{P}(A<900)=0.97$, find the value of μ and the value of σ. [6]

A bag of flour is deemed to be insufficient if it contains less than m grams of flour. It is known that, out of every 1000 bags, 15 are deemed insufficient.

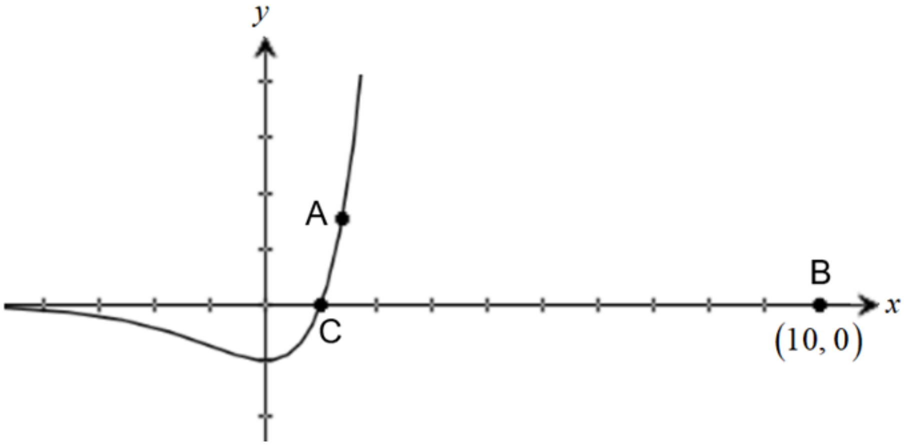
(e) Calculate the value of m. [3]

[exam continues on next page]

Do **not** write solutions on this page.

12. [Maximum mark: 16]

Consider the graph of the function $h(x) = (x-1)e^x$ shown below.



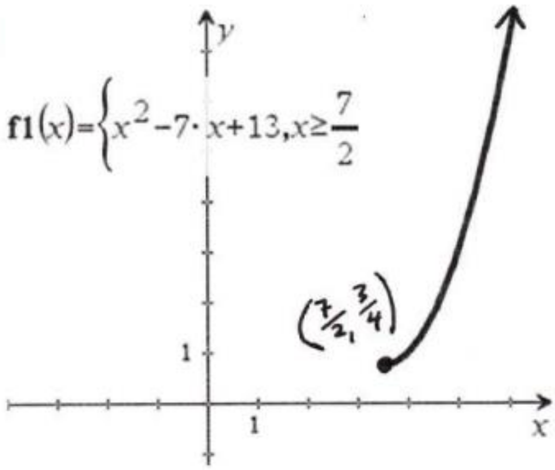
(a) Point C is the x-intercept of h. Find the equation of the line tangent to h at C. [4]

(b) Point A, with x-coordinate of a, lies on the graph h. Show that the equation, in terms of x and a, of the line that is tangent to h at point A is $y = ae^a x + e^a(-a^2 + a - 1)$. [3]

(c) Find the equation, in terms of x and a, of the line that is **normal** to h at point A. [4]

(d) Calculate the x-coordinate of the point on h that is closest to B. [5]

10. (a)



(b) range: $y \geq \frac{3}{4}$

(c) $y = x^2 - 7x + 13$

$x = y^2 - 7y + 13$ switch domain (x) and range (y)

$y^2 - 7y + \frac{49}{4} + 13 - \frac{49}{4} = x$ solve for y by completing the square

$\left(y - \frac{7}{2}\right)^2 + \frac{3}{4} = x$

$\left(y - \frac{7}{2}\right)^2 = x - \frac{3}{4}$

$y - \frac{7}{2} = \sqrt{x - \frac{3}{4}}$ square root of both sides; only positive square root because range of $f^{-1}(x)$ is $y \geq \frac{7}{2}$ (domain of $f(x)$)

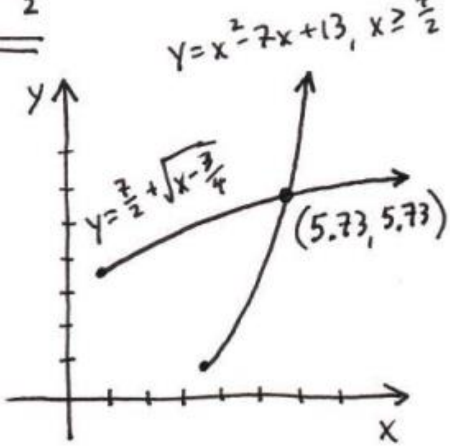
$f^{-1}(x) = \frac{7}{2} + \sqrt{x - \frac{3}{4}}$

(d) $f^{-1}(x)$ domain: $x \geq \frac{3}{4}$; range $y \geq \frac{7}{2}$

(e) $f(x) = f^{-1}(x)$

$x^2 - 7x + 13 = \frac{7}{2} + \sqrt{x - \frac{3}{4}}$, $x \geq \frac{7}{2}$

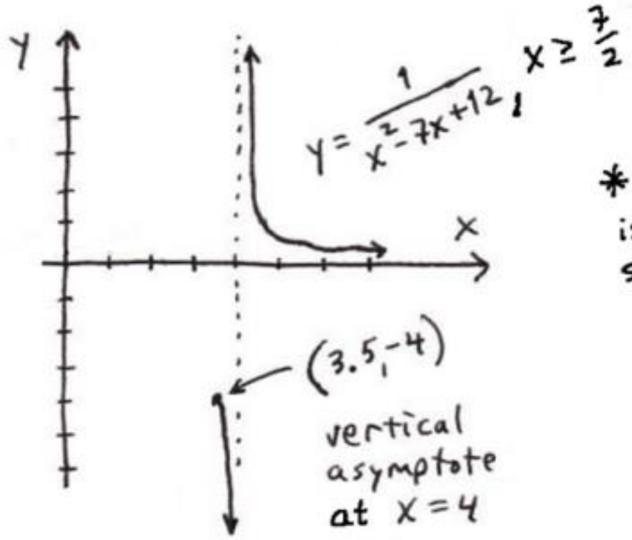
$x \approx 5.73$ $\left[x = 4 + \sqrt{3}\right]$



[Q10 worked solution continued on next page]

10. *[continued]*

(f) $g(f(x)) = \dfrac{1}{x^2 - 7x + 13 - 1} = \dfrac{1}{x^2 - 7x + 12} = \dfrac{1}{(x-4)(x-3)}$



* <u>note</u>: domain of f is $x \geq 7/2$ as given at start of question

(g) domain: $\frac{7}{2} \leq x < 4$, $x > 4$

(h) range: $\underline{\underline{y \leq \frac{7}{2},\ y > 0}}$

[Q11 worked solution on next page]

11. $X \sim N(748, 2.4^2)$ $\mu = 748, \ \sigma = 2.4$

(a) $P(X > 750) \approx 0.202328\ldots$ $P(X > 750) \approx 0.202$ Q.E.D.

(b) $Y \sim B(12, 0.202\ldots)$ $n = 12, \ p = 0.202\ldots$

$$P(Y=4) = \binom{12}{4}(0.202\ldots)^4(1-0.202\ldots)^8 \approx 0.135964\ldots$$

$$P(Y=4) \approx 0.136$$

(c) set up a table showing cumulative binomial probabilities starting at $x = 1$ and going to a large value, e.g $x = 1000$

$$P(Y \geq 1) = \sum_{x=1}^{1000} \binom{n}{x}(0.202\ldots)^x(1-0.202\ldots)^{n-x}$$

on GDC: $$y = \sum_{r=1}^{1000} \binom{x}{r}(0.202\ldots)^r(1-0.202\ldots)^{x-r}$$

OR use cumulative binomial probability command

table: $x = 15, \ y \approx 0.966321\ldots$
$x = 16, \ y \approx 0.973135\ldots$
$x = 17, \ y \approx 0.978571\ldots$
$x = 18, \ y \approx 0.982906\ldots$

therefore, minimum # of bottles is 18, i.e. $n = 18$

[Q11 worked solution continued on next page]

11. *[continued]*

(d) Find Z values for corresponding probabilities using GDC:

$$\mathrm{P}(A<850)=0.09 \quad \Rightarrow \quad Z=-1.3407\ldots$$

$$\mathrm{P}(A<900)=0.97 \quad \Rightarrow \quad Z=1.8807\ldots$$

Using the formula for standardized normal variable $Z=\dfrac{x-\mu}{\sigma}$:

$$-1.3407\ldots=\frac{850-\mu}{\sigma} \quad \Rightarrow \quad \mu-(1.3407\ldots)\sigma=850$$

$$1.8807\ldots=\frac{900-\mu}{\sigma} \quad \Rightarrow \quad \mu+(1.8807\ldots)\sigma=900$$

Solving system of linear equations on GDC:

$\mu \approx 871\mathrm{g},\ \sigma \approx 15.5\mathrm{g}$

(e) 15 out of 1000 bags are insufficient, so $\mathrm{P}(A<m)=\dfrac{15}{1000}=0.015$

Solving for m on GDC using inverse normal command with μ and σ from part (d):

$m \approx 837\,\mathrm{g}$

[Q12 worked solution on next page]

12. (a) $h(x) = (x-1)e^x$

$$h'(x) = e^x + (x-1)e^x = e^x\left[1+(x-1)\right] = xe^x$$

find x-coordinate of x-intercept C

$$(x-1)e^x = 0 \Rightarrow x = 1 \Rightarrow C(1,0)$$

$$h'(1) = e$$

tangent at C: $y = e(x-1) \Rightarrow \underline{\underline{y = ex - e}}$

(b) $h(a) = (a-1)e^a = ae^a - e^a \Rightarrow A\left(a, ae^a - e^a\right)$

$h'(a) = ae^a$ gradient of tangent at A

tangent at A: $y - (ae^a - e^a) = ae^a(x-a)$

$$y - ae^a + e^a = ae^a x - a^2e^a \Rightarrow y = ae^a x - a^2 e^a + ae^a - e^a$$

$$y = ae^a x + e^a(-a^2 + a - 1) \quad \underline{\text{Q.E.D.}}$$

(c) gradient of normal at A is $-\dfrac{1}{ae^a}$

normal at A: $y - (ae^a - e^a) = -\dfrac{1}{ae^a}(x-a)$

$$y - ae^a + e^a = -\frac{x}{ae^a} + \frac{1}{e^a} \Rightarrow \underline{\underline{y = -\frac{x}{ae^a} + \frac{1}{e^a} + ae^a - e^a}}$$

(d) the line segment of shortest distance from B(10,0) to graph of h must be normal to the graph; thus, find value of a such that the normal found in (c) passes through B(10,0).

solve: $0 = -\dfrac{10}{ae^a} + \dfrac{1}{e^a} + ae^a - e^a \Rightarrow a \approx 1.38727\ldots$

therefore, x-coordinate of point on h closest to B is approx. $\underline{\underline{1.39}}$